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Uniform Polynomial Approximation of Analytic Functions on a Quasidisk*

VLADIMIR ANDRIEVSKII

Institute for Applied Mathematics and Mechanics, Ukrainian Academy of Sciences, Donetsk 340114, Ukraine

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Let L be an arbitrary quasidisk, and f analytic in G and continuous on $\bar{G}$. We prove two theorems establishing a connection between the sequence of values $E_n(f, \bar{G})$, $n = 1, 2, \ldots$, of best uniform polynomial approximations of the function f on $\bar{G}$ and its smoothness properties on the boundary ∂G. Then we apply one of these results to the solution of a problem suggested by Turan concerning the correlation between polynomial and rational approximations on the unit disk.

1. INTRODUCTION

This paper is connected with the study of the values $E_n\ (f, \bar{G})$, $n = 0, 1, 2, \ldots$, of best uniform polynomial approximations of a function f analytic in a bounded Jordan domain $G \subset \mathbb{C}$ and continuous on its closure $\bar{G}$.

The rate of decrease of $E_n(f, \bar{G})$ as $n \to \infty$, the geometric structure of the boundary ∂G of G, and the smoothness of the function f near the boundary interact in a complicated way.

The main subject of our paper is the consideration of the following two problems.

Let $\mu(\delta)$, $\delta > 0$, be a so-called normal majorant (for example, $\mu(\delta) = \delta^c$, $c = \text{const} > 0$).

PROBLEM A. *Describe all functions f satisfying*

$$E_n(f, \bar{G}) = O(\mu(1/n)) \qquad \text{as} \quad n \to \infty. \tag{1.1}$$

PROBLEM B. *Describe all functions f for which*

$$E_n(f, \bar{G}) \sim \mu(1/n), \qquad n = 1, 2, \ldots, \tag{1.2}$$

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where the symbol $g \sim \varphi$ *means that*

$$1/c \leqslant g\varphi \leqslant c$$

holds for some constant $c > 0$.

Problem A is a typical problem in approximation theory. One can find a complete survey of results obtained in this direction in [6–9, 3].

Problem B is initiated by some similar results of Stechkin [12] concerning the approximation of real functions.

We give the solution of Problems A and B in the case when G is an arbitrary quasidisk [1] and apply then these results to the study of a problem of Turan concerning the correlation between polynomial and rational approximations on the unit disk.

2. Definitions and Main Results

Let K be an arbitrary compact set in the complex plane $\mathbb{C}$. We denote by $A(K)$ the class of all functions continuous on K and analytic in its interior. Let $\mathbb{P}_n$, $n = 0, 1, ...,$ be the class of all polynomials of degree at most n. For $f \in A(K)$, $z \in \mathbb{C}$, $\delta > 0$, $n = 0, 1, ...,$ and an integer $m \geqslant 1$ put

$$\|f\|_K := \sup\{|f(z)|, z \in K\},$$

$$E_n(f, K) := \inf\{\|f - p\|_K, p \in \mathbb{P}_n\},$$

$$D(z, \delta) := \{\zeta: |\zeta - z| < \delta\}, \qquad D := D(0, 1),$$

$$d(z, K) := \inf\{|\zeta - z|, \zeta \in K\},$$

$$\omega_{m, K}(f, z, \delta) := E_{m-1}(f, K \cap \overline{D(z, \delta)}).$$

Let $G \subset \mathbb{C}$ be an arbitrary bounded quasidisk [1] with complement $\Omega := \overline{\mathbb{C}} \backslash K$, and let $L = \partial G = \partial \Omega$ be their common boundary (hence L is a quasicircle). We recall that a geometric test for the quasiconformality of L is as follows: L is a quasicircle if and only if it is a Jordan curve and there exists a constant $c \geqslant 1$ such that for each pair of points $z_1, z_2 \in L$

$$\min_{j=1,2} \operatorname{diam}(\gamma_j) \leqslant c\,|z_1 - z_2|,$$

where γ_1 and γ_2 are the components of $L \backslash \{z_1, z_2\}$.

We denote by $w = \Phi(z)$ the function that maps Ω conformally onto $\Delta := \{w: |w| > 1\}$ with the normalization $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$. We extend Φ continuously on $\bar{\Omega}$, retaining the notation Φ for the extended function, and denote the inverse function by $\Psi := \Phi^{-1}$.

For an integer $m \geqslant 1$ we consider the following characteristic of the smoothness properties of a function f on L or, more exactly, of the function $\tilde{f}(w) := f[\Psi(w)]$ on $\{w : |w| = 1\} = \partial\Delta$:

$$\tilde{\omega}_m(\delta) := \sup\{E_{m-1}(f, \Psi(\gamma)), \gamma \subset \partial\Delta, |\gamma| \leqslant \delta\}, \qquad \delta > 0,$$

where $\gamma \subset \partial\Delta$ is an arbitrary arc, $|\gamma|$ is its length.

When $G = D$, $\tilde{\omega}_m(\delta)$ is equivalent to the mth modulus of continuity of the function f on ∂G (see more precisely [13, 6]).

We note also that $\tilde{\omega}_1(\delta)$ is simply equivalent to the usual modulus of continuity of a function $\tilde{f}$ on $\partial\Delta$.

We use $c, c_1, \ldots$ to denote positive constants (in general, different in different relations), depending, unless the contrary is explicitly stated, only on G or other inessential quantities.

Following [13], we call a function $\mu(\delta)$ a normal majorant if it is defined, finite, positive, and nondecreasing for $\delta > 0$ and satisfies

$$\mu(t\delta) \leqslant c_1 t^c \mu(\delta), \qquad t \geqslant 1, \quad \delta > 0. \tag{2.1}$$

For example, the function $\mu(\delta) = c_1 \delta^c$ is a normal majorant.

For convenience of formulation of our results we assume, without loss of generality, that

$$\lim_{\delta \to +0} \mu(\delta) = 0;$$

$$\mu(\delta) = c_1 \qquad \text{for} \quad \delta > c_2.$$

THEOREM 1. *Let G be a quasidisk, μ a normal majorant, $f \in A(\bar{G})$. In order that* (1.1) *holds it is necessary for all sufficiently large $m \geqslant m_0(\mu, G)$ and sufficient for some $m \geqslant 1$ that*

$$\tilde{\omega}_m(\delta) = 0(\mu(\delta)) \qquad as \quad \delta \to 0.$$

Remark. According to Theorem 1 and Lemma 3 inequality

$$E_n(f, \bar{G}) \leqslant c\tilde{\omega}_m(1/n), \qquad n = 1, 2, \ldots \tag{2.2}$$

holds for all integers $m \geqslant 1$, where $c = c(G, m)$.

In the majority of known results of this kind the particular case of (2.2) for $m = 1$ is most popular.

Unfortunately, the example of function $f(z) = z$ and domain

$$G = G_\alpha := \{z = re^{i\theta\pi} : 0 < r < 1, \alpha/2 < \theta < 2\}, \qquad 0 < \alpha < 1,$$

shows that even the condition $E_n(f, \bar{G}) = 0$ for $n \geqslant 1$ is not sufficient in order to assert that $\tilde{\omega}_1(\delta) = O(\delta^\alpha)$ as $\delta \to 0$.

This fact, in particular, explains the role of the quantity $\tilde{\omega}_m(\delta)$ because the transition from $m = 1$ to an arbitrary $m \geqslant 1$ gives us the possibility to obtain the description of functions with property (1.1).

THEOREM 2. *Let G be a quasidisk, μ a normal majorant, $f \in A(\bar{G})$. In order that* (1.2) *holds it is necessary and sufficient that*

$$\tilde{\omega}_m(\delta) \sim \mu(\delta), \qquad \delta > 0 \tag{2.3}$$

holds for all sufficiently large $m \geqslant m_0(\mu, G)$.

Theorem 2 can be applied to the solution of the following problem of Turan [14, Problem LXXXVII; 11, p. 363].

Denote by B the class of all functions $f \in A(\bar{D})$ which cannot be continued analytically beyond ∂D. Let $\rho_n(f, \bar{D})$, $n = 0, 1, \ldots$, be the best uniform approximation of the function f on $\bar{D}$ by rational functions of the form $R_n(z) := p_n(z)/q_n(z)$, where $p_n, q_n \in \mathbb{P}_n$.

Turan has asked whether it is true that there is no $f_0 \in B$ such that $E_n(f_0, \bar{D}) \geqslant c_1/n$, but $\rho_n(f_0, \bar{D}) \leqslant \exp\{-c_2 n^{1/2}\}$ for $n = 1, 2, \ldots$.

We give the negative answer on this question and even prove a stronger result.

THEOREM 3. *For any α: $0 < \alpha < 1$ there is a function $g = g_\alpha \in B$ such that*

$$E_n(g, \bar{D}) \geqslant c_1 n^{-\alpha}, \tag{2.4}$$

$$\rho_n(g, \bar{D}) \leqslant \exp\{-c_2 n^{1/2}\}, \tag{2.5}$$

where $c_i = c_i(\alpha)$, $i = 1, 2$.

We note that similar problems of rational approximation of analytic functions on compact sets of the complex plane were studied in [10]. There one can also find a survey of such results.

We use the notation $a \ll b$ to denote that $a \leqslant cb$.

3. LOCAL PROPERTIES OF THE CONFORMAL MAPPINGS Φ AND Ψ

In this section we recall some results from [1, 2, 4] that will be needed below.

Mappings Φ and Ψ can be extended to quasiconformal mappings of the whole complex plane on itself. Consequently, according to [2, Lemma 1] we can formulate the following assertion.

LEMMA 1. *For any three points $\zeta_j \in \bar{\Omega}$, $j=1,2,3$, the conditions $|\zeta_1-\zeta_2| \ll |\zeta_1-\zeta_3|$ and $|w_1-w_2| \ll |w_1-w_3|$, where $w_j := \Phi(\zeta_j)$, $j=1,2,3$, are equivalent and provide the inequalities*

$$\left|\frac{w_1-w_3}{w_1-w_2}\right|^{\alpha} \ll \left|\frac{\zeta_1-\zeta_3}{\zeta_1-\zeta_2}\right| \ll \left|\frac{w_1-w_3}{w_1-w_2}\right|^{\beta}$$

with some constants $\beta > \alpha > 0$ depending only on L.

In the following we use without proof some geometrical facts which follow easily from Lemma 1. We formulate one of them (see, for example, [3, Lemma 2]).

For arbitrary $u>0$ and $z \in \mathbb{C}$ we put

$$L_{1+u} := \{\zeta : |\Phi(\zeta)| = 1+u\},$$
$$\rho_u(z) := d(z, L_{1+u}).$$

LEMMA 2. *Let $u > v > 0$. Then for $z \in L$*

$$\left(\frac{u}{v}\right)^{\alpha} \ll \frac{\rho_u(z)}{\rho_v(z)} \ll \left(\frac{u}{v}\right)^{\beta}$$

holds, where α and β are constants from Lemma 1.

4. PROOF OF THEOREM 1

To begin with let us establish the following assertion.

LEMMA 3. *For all $f \in A(\bar{G})$ and $m \geq 1$ the function $\tilde{\omega}_m(\delta)$ is a normal majorant.*

Proof. It is obvious that the fulfillment of the condition (2.1) is only nontrivial. Let $0 < \delta < 2\pi$, $t \geq 1$, and let $\gamma \subset \partial D$ be an arc for which

$$E_{m-1}(f, \Psi(\gamma)) = \tilde{\omega}_m(t\delta), \qquad |\gamma| \leq t\delta.$$

Without loss of generality we can assume that $|\gamma| > \delta$. Denote by γ_1, $\gamma_2, ..., \gamma_k$ the system of arcs with the following properties:

(i) $\gamma = \bigcup_{j=1}^{k} \gamma_j$;

(ii) $\delta/2 \leq |\gamma_j| \leq \delta$, $j=\overline{1,k}$;

(iii) $|\gamma_j \cap \gamma_{j+1}| \geq \delta/2$, $j = \overline{1, k-1}$;

(iv) $k \ll t$.

We put $l := \Psi(\gamma)$, $l_j := \Psi(\gamma_j)$, $j = \overline{1, k}$. Choose j and let $p_j(z) = p_j(z, f, m) \in \mathbb{P}_{m-1}$ be a polynomial such that

$$\|f - p_j\|_{l_j} = E_{m-1}(f, l_j).$$

Consider the polynomial $q_j(z) := p_{j+1}(z) - p_j(z)$. For $z \in l_j \cap l_{j+1}$ we have

$$|q_j(z)| \leqslant |p_j(z) - f(z)| + |f(z) - p_{j+1}(z)| \leqslant 2\tilde{\omega}_m(\delta). \tag{4.1}$$

On arc $\gamma_j \cap \gamma_{j+1}$ one can construct the system of points $\omega_1, ..., \omega_m$ according to the following rule.

(i) If $m = 1$, then $\omega_1 \in \gamma_j \cap \gamma_{j+1}$ is an arbitrary point.

(ii) If $m > 1$, then ω_1 and ω_m are end points of the arc $\gamma_j \cap \gamma_{j+1}$ and the other points are defined by

$$|\omega_i - \omega_{i+1}| = 2 \sin \frac{|\gamma_j \cap \gamma_{j+1}|}{2(m-1)}, \qquad i = \overline{1, m-1}.$$

By Lemma 1 the set of points $z_i := \Psi(\omega_i)$ satisfies for all i, $s = \overline{1, m}$, $i \neq s$

$$|z_i - z_s| \sim \operatorname{diam} l_j \sim \operatorname{diam} l_{j+1},$$

and for each point $z \in l$

$$\left| \frac{z - z_s}{z_i - z_s} \right| \ll 1 + \left| \frac{\Phi(z) - \omega_s}{\omega_i - \omega_s} \right|^{\beta} \ll t^{\beta}.$$

By virtue of inequality (4.1) and the Lagranges interpolation formula

$$q_j(z) = \sum_{i=1}^{m} q_j(z_i) \frac{\pi_i(z)}{\pi_i(z_i)}, \qquad \pi_i(z) = \prod_{\substack{s=1 \\ s \neq i}}^{m} (z - z_s)$$

we successively obtain for $z \in l$

$$|q_j(z)| \ll \tilde{\omega}_m(\delta) \sum_{i=1}^{m} t^{\beta(m-1)} \sim t^{\beta(m-1)} \tilde{\omega}_m(\delta).$$

Therefore, if $z \in l_j$, then

$$|f(z) - p_1(z)| \leqslant |f(z) - p_j(z)| + \sum_{i=1}^{j-1} |q_i(z)| \ll j t^{\beta(m-1)} \tilde{\omega}_m(\delta)$$

$$\ll t^{\beta(m-1)+1} \tilde{\omega}_m(\delta). \tag{4.2}$$

Consequently,

$$\tilde{\omega}_m(t\delta) = E_{m-1}(f, l) \leqslant \|f - p_1\|_l \ll t^{\beta(m-1)+1}\tilde{\omega}_m(\delta),$$

i.e., inequality (2.1) is satisfied for the function $\tilde{\omega}_m(\delta)$ with $c = \beta(m-1)+1$.

Now let $r(z, h)$, $z \in L$, $h \geqslant 0$, be a function defined by the identity

$$\rho_{r(z,h)}(z) = h.$$

Let $z \in L$ be an arbitrary point, $w := \Phi(z)$, and let $\gamma \subset \partial\Delta$ be an arc such that $w \in \gamma$, $|\gamma| = h$, $0 < h < 2\pi$. Denote by $p_0 \in \mathbb{P}_{m-1}$ the polynomial for which

$$\|f - p_0\|_{\Psi(\gamma)} = E_{m-1}(f, \Psi(\gamma)).$$

Reasoning like in the proof of inequality (4.2) one can obtain for $\zeta \in L$

$$|f(\zeta) - p_0(\zeta)| \ll \begin{cases} \tilde{\omega}_m[r(z, h)], & |\zeta - z| \leqslant h; \\ \tilde{\omega}_m[r(z, h)]\left[\left(\dfrac{\Phi(\zeta) - \Phi(z)}{r(z, h)}\right)\right]^c, & |\zeta - z| > h. \end{cases} \tag{4.3}$$

Using in the case $|\zeta - z| > h$ the estimates

$$\left|\frac{\Phi(\zeta) - \Phi(z)}{r(z, h)}\right| \sim \left|\frac{\Phi(\zeta) - \Phi(z)}{\Phi(z_h) - \Phi(z)}\right| \ll \left|\frac{\zeta - z}{h}\right|^{1/\alpha},$$

where z_h is an arbitrary point of the intersection $\partial D(z, h) \cap \Omega$ we can write (4.3) in the form

$$|f(\zeta) - p_0(\zeta)| \ll \tilde{\omega}_m[r(z, h)]\left(1 + \left|\frac{z - z_0}{h}\right|^{c/\alpha}\right), \qquad \zeta \in L.$$

By a result of Tamrazov (see, for example, [7, p. 425]) we have

$$\omega_{m,G}(f, z, h) := E_{m-1}(f, \overline{D(z, h) \cap G}) \ll \tilde{\omega}_m[r(z, h)]. \tag{4.4}$$

To complete the proof of Theorem 1 it is sufficient to use a slightly modified version of the description of function classes with property (1.1) suggested in [3].

We confine ourselves to the formulation of this assertion for quasidisks only.

LEMMA 4. *Let G be a quasidisk, μ a normal majorant, $f \in A(\overline{G})$. In order that* (1.1) *holds it is necessary for all sufficiently large $m \geqslant m_0(\mu, G)$ and sufficient for some $m \geqslant 1$ that*

$$\omega_{m,G}(f, z, h) \ll \mu[r(z, h)]$$

holds for all $z \in L$ and $h > 0$.

5. Proof of Theorem 2

We have to establish for sufficiently large m the equivalence of the following two double inequalities

$$c_1 \mu(1/n) \leqslant E_n(f, \bar{G}) \leqslant c_2 \mu(1/n), \qquad n = 1, 2, ..., \tag{5.1}$$

$$c_3 \mu(\delta) \leqslant \tilde{\omega}_m(\delta) \leqslant c_4 \mu(\delta), \qquad \delta > 0. \tag{5.2}$$

Let (5.1) be true. The correctness of the right-hand part of (5.2) for sufficiently large m follows from Theorem 1. For $0 < \delta < 1$, choosing integer n such that $(n+1)^{-1} \leqslant \delta < n^{-1}$, we have by Lemma 3 and inequality (2.2)

$$\mu(\delta) \leqslant \mu(1/n) \ll E_n(f, \bar{G}) \ll \tilde{\omega}_m(1/n) \ll \tilde{\omega}_m(1/(n+1)) \leqslant \tilde{\omega}_m(\delta).$$

Hence the correctness of the left-hand part of (5.2) is proved (even for all $m \geqslant 1$).

Now let (5.2) be satisfied. The right-hand part of the inequality (5.1) follows from Theorem 1. Let us verify the correctness of the left-hand part of this estimate.

Let $P_n \in \mathbb{P}_n$, $n = 0, 1, ...$, be such that

$$\|f - P_n\|_{\bar{G}} = E_n(f, \bar{G}).$$

Lemma 5. *If*

$$E_n(f, \bar{G}) \ll \mu(1/n), \qquad n = 1, 2, ..., \tag{5.3}$$

then for sufficiently large $m \geqslant m_0(\mu, G)$, $z_0 \in L$, $z \in D(z_0, \rho_0/2)$, *where* $\rho_0 := \rho_{1/n}(z_0)$, *and* $0 < \delta < \rho_0/2$

$$|P_n^{(m)}(z)| \ll \rho_0^{\ -m} \mu(1/n), \tag{5.4}$$

$$\omega_{m, \bar{G}}(P_n, z_0, \delta) \ll \mu(1/n)(\delta/\rho_0)^m. \tag{5.5}$$

Proof. Choose an integer s such that $2^s \leqslant n < 2^{s+1}$. The polynomial P_n can be rewritten in the form

$$P_n(z) = \sum_{j=0}^{s+1} Q_j(z),$$

where

$$Q_j(z) := \begin{cases} P_1(z), & j = 0; \\ P_{2^j}(z) - P_{2^{j-1}}(z), & 1 \leqslant j \leqslant s; \\ P_n(z) - P_{2^s}(z), & j = s+1. \end{cases}$$

According to (5.3) for polynomials $Q_j(z)$ we have $\|Q_j\|_{\bar{G}} \ll \mu(2^{1-j})$, $1 \leqslant j \leqslant s+1$. By the Bernstein–Walsh theorem [15, p. 77] we have

$$\|Q_j\|_{\overline{D(z_0, \rho_j)}} \ll \mu(2^{-j}),$$

where $\rho_j := \rho_{2^{-j}}(z_0)$. Consequently for $z \in D(z_0, \rho_0/2)$ we obtain

$$\|Q_j^{(m)}(z)| \leqslant \frac{m!}{2\pi} \int_{\partial D(z_0, \rho_j)} \frac{|Q_j(\zeta)|}{|\zeta - z|^{m+1}} |d\zeta| \ll \mu(2^{-j}) \rho_j^{-m}.$$

Therefore, if $z \in D(z_0, \rho_0/2)$ and $m \geqslant 2$ we have by (2.1) and Lemma 2

$$\begin{aligned} |P_n^{(m)}(z)| &\leqslant \sum_{j=1}^{s+1} |Q_j^{(m)}(z)| \ll \sum_{j=1}^{s+1} \mu(2^{-j}) \rho_j^{-m} \\ &= \mu(2^{-s-1}) \rho_{s+1}^{-m} \sum_{j=1}^{s+1} \frac{\mu(2^{-j})}{\mu(2^{-s-1})} \left[\frac{\rho_{2^{-s-1}}(z_0)}{\rho_{2^{-j}}(z_0)}\right]^m \\ &\ll \mu(1/n) \rho_0^{-m} \sum_{j=1}^{s+1} 2^{(s-j)c} 2^{(j-s)\alpha m} \ll \mu(1/n) \rho_0^{-m}, \end{aligned}$$

as soon as $m > c/\alpha$.

Inequality (5.5) immediately follows from estimate (5.4). Indeed, since

$$P_n(z) = \sum_{j=0}^{m-1} \frac{1}{j!} (z - z_0)^j P_n^{(j)}(z_0) + \frac{1}{(m-1)!} \int_{[z_0, z]} (z - \zeta)^{m-1} P_n^{(m)}(\zeta)\, d\zeta$$

according to (5.4) we find

$$\begin{aligned} \omega_{m, \bar{G}}(P_n, z_0, \delta) &\leqslant \sup_{z \in D(z_0, \delta)} \left| P_n(z) - \sum_{j=0}^{m-1} \frac{1}{j!} (z - z_0)^j P_n^{(j)}(z_0) \right| \\ &\ll \mu(1/n)(\delta/\rho_0)^m. \end{aligned}$$

Thus Lemma 5 is proved.

Let $z_0 \in L$ be an arbitrary point. By our assumption (5.3) holds. Therefore, for polynomials $P_n(z)$, (5.5) is true and, consequently, by Lemma 5 we have

$$\begin{aligned} \omega_{m, \bar{G}}(f, z_0, \delta) &\leqslant \omega_{m, \bar{G}}(f - P_n, z_0, \delta) + \omega_{m, \bar{G}}(P_n, z_0, \delta) \\ &\leqslant E_n(f, \bar{G}) + c_1 \mu(1/n)(\delta/\rho_0)^m. \end{aligned}$$

Let $s > 1$ be such that $\delta := \rho_{1/(sn)}(z_0) < \rho_0/2$ (a more concrete choice of the number $s = s(\mu, G)$ will be specified below).

According to (2.1) and Lemma 2

$$c_1\,\mu(1/n)(\delta/\rho_0)^m \leqslant c_2\,\mu(1/(sn))\,s^c\left[\frac{\rho_{1/(sn)}(z_0)}{\rho_{1/n}(z_0)}\right]^m \leqslant c_3\,\mu(1/(sn))\,s^{c-\alpha m}.$$

Therefore

$$E_n(f, \bar{G}) \geqslant \omega_{m,L}(f, z_0, \delta) - c_3\,\mu(1/(sn))\,s^{c-\alpha m}. \tag{5.6}$$

By Lemma 1

$$\Psi(\gamma) \subset D(z, \rho_{c_4|\gamma|}(z))$$

holds for any arc $\gamma \subset \partial\Delta$ and point $z \in \gamma$.

Thus choosing in (5.6) point z_0 such that

$$\omega_{m,L}(f, z_0, \delta) \geqslant \tilde{\omega}_m[(snc_4)^{-1}]$$

we successively obtain for $m > c/\alpha$

$$\begin{aligned} E(f, \bar{G}) &\geqslant \tilde{\omega}_m[(snc_4)^{-1}] - c_3\,\mu[(sn)^{-1}]\,s^{c-\alpha m} \geqslant \mu[(sn)^{-1}](c_5 - c_3 s^{c-\alpha m}) \\ &\geqslant c_5\,\mu[(sn)^{-1}]/2 \geqslant c_6\,\mu(1/n) \end{aligned}$$

as soon as $s \geqslant (2c_3/c_5)^{1/(\alpha m - c)}$.

6. Proof of Theorem 3

Choose α: $0 < \alpha < 1$ and consider the function $f(z) = f_\alpha(z) := (z-1)^\alpha$. For this function

$$E_n(f, \bar{D}) \gg n^{-\alpha}, \qquad n = 1, 2, \ldots. \tag{6.1}$$

Indeed, according to Theorem 2 it is sufficient for the proof of (6.1) to establish the double inequality

$$\delta^\alpha \ll \tilde{\omega}_m(\delta) \ll \delta^\alpha \tag{6.2}$$

for $0 < \delta < 1$ and $m \geqslant 1$.

In fact, the right-hand part of (6.2) is obvious:

$$\tilde{\omega}_m(\delta) \leqslant \omega_{1,\bar{D}}(f, 1, \delta) = \delta^\alpha/2.$$

Now let $p \in \mathbb{P}_{m-1}$ be a polynomial for which

$$\omega_{m,\bar{D}}(f, 1, \delta) = \|f - p\|_{\overline{D \cap D(1,\delta)}}.$$

We have for $0<\delta<1$

$$\begin{aligned}|\alpha\cdot(\alpha-1)\cdots(\alpha-m+1)|\,(\delta/2)^{\alpha-m}&=|f^{(m)}(1-\delta/2)|\\&=\frac{m!}{2\pi}\left|\int_{\partial D(1-\delta/2,\,\delta/2)}\frac{f(\zeta)-p(\zeta)}{(\zeta+\delta/2-1)^{m+1}}\,d\zeta\right|\\&\leqslant m!\,(\delta/2)^{-m}\,\omega_{m,\bar D}(f,1,\delta).\end{aligned}$$

Consequently by (4.4)

$$\tilde\omega_m(\delta)\gg\omega_{m,\bar D}(f,1,\delta)\gg\delta^{\alpha}.$$

Thus the relation (6.1) is proved.

Now let us estimate from above the speed of rational approximation of the function f on $\bar D$.

According to the Cauchy formula

$$f(z)=\frac{1}{2\pi i}\int_{|\zeta|=2}\frac{f(\zeta)}{\zeta-z}\,d\zeta+\int_{[1,2]}\frac{h(\zeta)}{\zeta-z}\,d\zeta, \tag{6.3}$$

where $|z|<1$, $h(\zeta)=h_\alpha(\zeta):=(1/2\pi i)\,|\zeta-1|^{\alpha}\,(1-e^{2\pi\alpha i})$.

The first integral in (6.3) can be approximated even by polynomials with rate of geometrical progression. In constructing a rational approximation for the second integral choose q: $0<q<1$ and consider the system of points

$$\zeta_j:=1+q^j,\qquad j=0,1,2,\ldots.$$

Since for every integer k

$$\frac{1}{\zeta-z}=R_{k,j}(\zeta,z)+\left(\frac{\zeta_j-\zeta}{\zeta_j-z}\right)^k\frac{1}{\zeta-z},$$

$$\text{where}\quad R_{k,j}(\zeta,z):=\sum_{i=0}^{k-1}\frac{(\zeta_j-\zeta)^i}{(\zeta_j-z)^{i+1}},$$

for $\zeta\in[\zeta_{j+1},\zeta_j]$ and $z\in D$ we derive

$$\left|\frac{1}{\zeta-z}-R_{k,j}(\zeta,z)\right|=\left|\frac{\zeta_j-\zeta}{\zeta_j-z}\right|^k\frac{1}{|\zeta-z|}\leqslant\frac{(1-q)^k}{|\zeta-z|}.$$

For some integers s and k consider the rational function

$$R(z):=\sum_{j=0}^{s-1}\int_{[\zeta_{j+1},\zeta_j]}h(\zeta)\,R_{k,j}(\zeta,z)\,d\zeta$$

of degree at most sk.

It is easy to see that

$$\left|\int_{[1,2]} \frac{h(\zeta)}{\zeta - z}\,d\zeta - R(z)\right| \leqslant \int_{[1,\zeta_s]} \frac{|h(\zeta)|}{|\zeta - z|}\,|d\zeta| + \sum_{j=0}^{s-1} \int_{[\zeta_{j+1},\zeta_j]} |h(\zeta)| \times \left|\frac{1}{\zeta - z} - R_{k,j}(\zeta, z)\right| |d\zeta| \ll q^{\alpha s} + (1-q)^k.$$

Therefore, if we put $s = k = [n^{1/2}]$, then the last inequality allows to write for the function f

$$\rho_n(f, \bar{D}) \ll \exp\{-c_1 n^{1/2}\}.$$

Now consider the function

$$\varphi(z) := \sum_{k=0}^{\infty} z^{2^k} \exp\{-2^{k/2}\}.$$

By the Hadamard theorem [5, p. 42], $\varphi \in B$. Let n be arbitrary, and let the integer j be defined by $2^j \leqslant n < 2^{j+1}$. We find

$$\rho_n(\varphi, \bar{D}) \leqslant E_n(\varphi, \bar{D}) \leqslant \max_{z\in D} \left|\sum_{k=j+1}^{\infty} z^{2^k} \exp\{-2^{k/2}\}\right| \leqslant \sum_{k=j+1}^{\infty} \exp\{-2^{k/2}\} \ll \exp\{-n^{1/2}\}.$$

It is easy to see that the function $g := f + \varphi$ satisfies (2.4) and (2.5).

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